

ON THE BEST APPROXIMATION OF THE INFINITESIMAL GENERATOR OF A CONTRACTION SEMIGROUP IN A HILBERT SPACE¹

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Abstract: Let A be the infinitesimal generator of a strongly continuous contraction semigroup in a Hilbert space H . We give an upper estimate for the best approximation of the operator A by bounded linear operators with a prescribed norm in the space H on the class $Q_2 = \{x \in \mathcal{D}(A^2) : \|A^2x\| \leq 1\}$, where $\mathcal{D}(A^2)$ denotes the domain of A^2 .

Key words: Contraction semigroup, Infinitesimal generator, Stechkin's problem.

1. Introduction

Let H be a Hilbert space with the inner product (\cdot, \cdot) and the norm $\|\cdot\|$, and let A be the infinitesimal generator of a strongly continuous contraction semigroup in H . For the definition and properties of the infinitesimal generator of a semigroup in a Banach space see, e.g., [6, §14.2]. Note that a strongly continuous contraction semigroup is also called a contraction semigroup of the class C_0 ([8, 9]). For an operator F on the space H , $\mathcal{D}(F)$ denotes the domain of F . We denote by I the identity operator.

In this paper, we study the so-called Stechkin's problem of the best approximation of the operator A by bounded linear operators with a prescribed norm on the class of elements $x \in \mathcal{D}(A^2)$ such that $\|A^2x\| \leq 1$. We give an upper estimate for the best approximation of the operator A .

The problem we consider is a special case of the general problem of the best approximation of an unbounded operator by linear bounded ones on a certain class of elements in a Banach space. This problem first appeared in Stechkin's work in 1965–1967 [11]. The problem was studied by a number of authors (see surveys [1], [2], monograph [4], paper [3], and the bibliography therein).

Stechkin formulated this problem in a general setting as follows. Let X, Y be two Banach spaces, let A be a linear operator (in general, unbounded) from X to Y , and let $Q \subseteq \mathcal{D}(A)$ be a certain class of elements from the domain $\mathcal{D}(A)$ of the operator A . We denote by $\mathcal{B}(N)$ the set of

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linear bounded operators from X to Y with the norm $\|T\|_{X \rightarrow Y} \leq N$. The best approximation of the operator A by linear bounded operators $T \in \mathcal{B}(N)$ on the class Q is

$$E_N(A; Q) = \inf \{U(A, T, Q) : T \in \mathcal{B}(N)\},$$

where

$$U(A, T, Q) = \sup \{\|Ax - Tx\|_Y : x \in Q\}$$

is the deviation of the operator T from the operator A on the class Q .

One of the most important cases of the problem formulated above is when the class Q is defined in the following way. Let Z be a Banach space and B be a linear operator from X to Z such that $\mathcal{D}(B) \subseteq \mathcal{D}(A)$. The class Q is then defined as $Q = \{x \in X : \|Bx\|_Z \leq 1\}$.

Stechkin [11] suggested an estimate from below for the best approximation $E_N(A; Q)$ in terms of the modulus of continuity of the operator A on the class Q defined by

$$\Phi(\delta) = \sup \{\|Ax\|_Y : x \in Q, \|x\|_X \leq \delta\}, \quad \delta > 0.$$

Namely, Stechkin showed that

$$E_N(A; Q) \geq \sup \{\Phi(\delta) - N\delta : \delta > 0\}. \quad (1.1)$$

In particular, when $B = A^n$, the problem $E_N(A^k; Q)$ turned out to be closely connected to the exact constants in the Kolmogorov-type inequalities of the form

$$\|A^k x\| \leq C \|x\|^{\frac{n-k}{n}} \|A^n x\|^{\frac{k}{n}}, \quad x \in \mathcal{D}(A^n), \quad (1.2)$$

with $n, k \in \mathbb{N}$, $0 < k < n$, and a certain constant C that depends on n and k .

If A is the differentiation operator, inequalities (1.2) are inequalities between the norms of the derivatives of a function. Such inequalities have been studied by a large number of authors (see [1], [2], [4] and the bibliography therein). Here we only mention that Hardy, Littlewood and Pólya [7, Chapter VII, §7.8] obtained the exact inequality

$$\|f'\|^2 \leq 2\|f\|\|f''\| \quad (1.3)$$

in the space $L_2(0, \infty)$ on the class of functions $f \in L_2(0, \infty)$ such that f' is locally absolutely continuous on $(0, \infty)$, and $f'' \in L_2(0, \infty)$.

In 1971, Kato [9] proved the following result which can be considered as a generalization of (1.3). Let A be the infinitesimal generator of a strongly continuous contraction semigroup in a Hilbert space H . Then

$$\|Ax\|^2 \leq 2\|x\|\|A^2x\|, \quad x \in \mathcal{D}(A^2).$$

In this paper, we study Stechkin's problem of the best approximation of the infinitesimal generator A of a strongly continuous contraction semigroup by bounded linear operators on the class

$$Q_2 = \{x \in \mathcal{D}(A^2) : \|A^2x\| \leq 1\} \quad (1.4)$$

in a Hilbert space. Namely, we estimate

$$E_N(A; Q_2) = \inf \{U(T) : T \in \mathcal{B}(N)\}, \quad (1.5)$$

where

$$U(T) = U(A, T, Q_2) = \sup \{\|Ax - Tx\| : x \in Q_2\}. \quad (1.6)$$

2. The main result

The main result of the paper is the following statement.

Theorem 1. *The best approximation (1.5) of the infinitesimal generator A of a strongly continuous contraction semigroup in a Hilbert space on the class Q_2 defined in (1.4) satisfies the inequality*

$$E_N(A; Q_2) \leq \frac{1}{N}.$$

It is known that the infinitesimal generator A of a strongly continuous contraction semigroup in a Banach space possesses the following properties:

- 1) The domain $\mathcal{D}(A)$ of the operator A is dense (see, e.g., [6, Lemma 14.5, p. 411]).
- 2) The resolvent set $\rho(A)$ of the operator A contains the right half-plane $\{\lambda \in \mathbb{C} \mid \Re \lambda > 0\}$. Moreover, $\|(A - \lambda I)^{-1}\| \leq (\Re \lambda)^{-1}$ for all $\lambda \in \mathbb{C}$ with $\Re \lambda > 0$ (e.g., [6, Theorem 14.7, p. 412]).

Furthermore, if A is the infinitesimal generator of a strongly continuous contraction semigroup in a Hilbert space, we have additionally:

- 3) The operator A is upper semibounded, with the upper bound 0, i.e.,

$$\Re(Ax, x) \leq 0$$

for $x \in \mathcal{D}(A)$ [6, Lemma 14.9, p. 416].

The following lemma is not new. However, we will formulate and prove it for the sake of completeness.

Lemma 1. *Let A be the infinitesimal generator of a strongly continuous contraction semigroup in a Hilbert space H and $c > 0$. Then the operator*

$$B_c = (cI + A)(cI - A)^{-1}$$

is densely defined and bounded (and thus can be extended to the whole space H by continuity). Moreover,

$$\|B_c\| \leq 1.$$

Remark. The operator B_c is the Cayley transform of the operator A in the terminology of Kato [9], see also [10, p. 545].

P r o o f. Since $c > 0$, the operator $(cI - A)^{-1}$ is defined everywhere on H and bounded. Since A is the infinitesimal generator of a strongly continuous contraction semigroup, the operator $-A$ is m -accretive (see [10, Chapter IX, §1.4 as well as Problem 1.18, both p. 485]). Therefore, the domain $\mathcal{D}(A)$ of the operator A is equal to the range $\mathcal{R}((cI - A)^{-1})$ of the operator $(cI - A)^{-1}$ which is dense in H (see [10, Chapter V, §3.10, p. 279]). Thus, B_c is densely defined.

Now we estimate the norm of B_c . For $x \in \mathcal{D}(A)$ we have

$$\|cx + Ax\|^2 = c^2\|x\|^2 + \|Ax\|^2 + 2c\Re(Ax, x),$$

$$\|cx - Ax\|^2 = c^2\|x\|^2 + \|Ax\|^2 - 2c\Re(Ax, x).$$

It follows immediately that

$$\|(cI + A)x\| \leq \|(cI - A)x\|. \quad (2.1)$$

Now take $y \in \mathcal{D}((cI - A)^{-1})$. Applying (2.1) to $x = (cI - A)^{-1}y \in \mathcal{D}(A)$, we obtain

$$\|(cI + A)(cI - A)^{-1}y\| \leq \|y\|,$$

and thus $\|B_c\| \leq 1$. □

Now we are ready to prove Theorem 1.

P r o o f. We will construct a concrete approximating operator T in problem (1.5) and estimate its norm and its deviation (1.6) from the operator A on the class Q_2 .

Note that all the operators we consider commute on the set $\mathcal{D}(A^2)$.

The restriction of the operator A to the set $\mathcal{D}(A^2)$ (which we will denote by the same symbol) can be represented as

$$A = \frac{N}{2}(B_N - I) - \frac{1}{2N}(B_N + I)A^2.$$

Put $T : H \rightarrow H$,

$$T = \frac{N}{2}(B_N - I).$$

Then, for the restriction of the operator $A - T$ to $\mathcal{D}(A^2)$, we have

$$A - T = -\frac{1}{2N}(B_N + I)A^2.$$

We estimate the norm of the operator T as follows:

$$\|T\| = \frac{N}{2}\|B_N - I\| \leq \frac{N}{2}(\|B_N\| + \|I\|) = N. \quad (2.2)$$

For the deviation $U(T)$ of the operator T from the operator A , we obtain that

$$U(T) = \sup_{x \in Q_2} \|(A - T)x\| \leq \sup_{x \in Q_2} \frac{1}{2N}\|B_N + I\| \cdot \|A^2x\| \leq \frac{1}{N}. \quad (2.3)$$

It follows immediately from (2.2) and (2.3) that

$$E_N(A; Q_2) \leq U(T) \leq \frac{1}{N}.$$

□

3. Approximation of the differentiation operator in the space $L_2(0, \infty)$

An important concrete case of problem (1.5) is the problem of the best approximation of the differentiation operator $Df = f'$ by bounded linear operators in the Hilbert space $L_2(0, \infty)$ of real-valued functions whose squares are integrable on $(0, \infty)$ on the class $Q^{(2)}$ defined as follows:

$Q^{(2)}$ is the class of functions $f \in L_2(0, \infty)$ such that f' is locally absolutely continuous on $[0, \infty)$, $f'' \in L_2(0, \infty)$, and $\|f''\| \leq 1$. Problem (1.5) takes in this case the form

$$E_N(D; Q^{(2)}) = \inf_{T \in \mathcal{B}(N)} \sup_{f \in Q^{(2)}} \|f' - Tf\|. \quad (3.1)$$

It took about 20 years of research to solve the problem completely. Stechkin's inequality (1.1) and inequality (1.3) of Hardy, Littlewood and Pólya provide the lower bound

$$E_N(D; Q^{(2)}) \geq \frac{1}{2N}.$$

One of the first upper bounds for (3.1)

$$E_N(D; Q^{(2)}) \leq \frac{1}{\sqrt{3}N}$$

was obtained by using a concrete approximating operator by the first named author in 1996 [5]. Problem (3.1) was fully solved only in 2014 by Arestov and the second named author [3]. Namely, they showed that

$$E_N(D; Q^{(2)}) = \frac{1}{2N}.$$

In this section, we discuss what the statement of Theorem 1 means in the concrete case (3.1) of problem (1.5). The approximating operator T used in Theorem 1 is

$$T = \frac{N}{2}(B_N - I) = NA(NI - A)^{-1}. \quad (3.2)$$

Below we will describe this operator in the special case. We consider and calculate its norm $\|T\|$ and its deviation $U(T)$ from the operator $A = D$ on the class $Q^{(2)}$.

It is not difficult to see that the operator T in the concrete case can be represented as follows. Let W be the class of functions $y \in L_2(0, \infty)$ such that y is locally absolutely continuous on $[0, \infty)$ and $y' \in L_2(0, \infty)$. For $f \in L_2(0, \infty)$, we consider the differential equation

$$-y' + Ny = f, \quad y \in W. \quad (3.3)$$

For each function $f \in L_2(0, \infty)$, equation (3.3) has a unique solution which is a real-valued function from $L_2(0, \infty)$. The operator T is defined as

$$Tf = Ny', \quad (3.4)$$

where y is the solution of the differential equation (3.3).

Integrating by parts and taking into account that $\lim_{t \rightarrow \infty} y(t) = 0$, we obtain (see [3] for details) that

$$\|f\|^2 = \int_0^\infty (-y'(t) + Ny(t))^2 dt = \int_0^\infty (y'(t))^2 dt + N^2 \int_0^\infty (y(t))^2 dt + Ny^2(0).$$

It follows from (3.4) that $\|Tf\|^2 = N^2 \int_0^\infty (y'(t))^2 dt$. Thus, we immediately obtain

$$\|Tf\|^2 \leq N^2 \|f\|^2, \quad (3.5)$$

which gives the estimate $\|T\| \leq N$. Now we show that indeed $\|T\| = N$. Consider the family of functions $y_K = e^{-Kt}$, $K > 0$. Let f_K be the corresponding right-hand side of equation (3.3). Take an arbitrary $0 < \alpha < 1$. We have

$$\begin{aligned} \alpha N^2 \|f_K\|^2 - \|Tf_K\|^2 &= \alpha N^2 \int_0^\infty (-y'_K(t) + Ny_K(t))^2 dt - N^2 \int_0^\infty (y'_K(t))^2 dt \\ &= \frac{N^2}{2K} (\alpha(K+N)^2 - K^2). \end{aligned}$$

This expression is negative for all $0 < \alpha < \frac{K^2}{(N+K)^2}$ which yields $\|Tf_K\|^2 > \alpha N^2 \|f_K\|^2$. Letting K go to infinity (with fixed N) we let α approach 1, and thus obtain $\|T\| \geq N$. Consequently, $\|T\| = N$.

Note that inequality (3.5) is a strict inequality if $y \neq 0$ and, consequently, $f \neq 0$. In other words, the norm of the operator T is not attained.

It can be shown similarly that the norm of the operator $V = -\frac{1}{2N}(B_N + I)$ is equal to $1/N$. Since the domain $\mathcal{D}(D^2)$ of the operator D^2 is dense in $L_2(0, \infty)$, it follows that the deviation of the operator T from the differentiation operator D on the class $Q^{(2)}$ is equal to $1/N$.

Thus, the approximating operator (3.2) gives the estimate $E_N(D; Q^{(2)}) \leq \frac{1}{N}$ in the general case (1.5) as well as in the concrete case (3.1).

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